The Weyl approach to the representation theory of reflection equation algebra

2004 J. Phys. A: Math. Gen. 37 5021

(http://iopscience.iop.org/0305-4470/37/18/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 5021–5046 PII: S0305-4470(04)70674-5

The Weyl approach to the representation theory of reflection equation algebra

P A Saponov

Theory Department, Institute for High Energy Physics, 142281 Protvino, Russia

E-mail: saponov@mx.ihep.su

Received 20 October 2003, in final form 18 March 2004
Published 20 April 2004
Online at stacks.iop.org/JPhysA/37/5021 (DOI: 10.1088/0305-4470/37/18/008)

Abstract
The present paper deals with the representation theory of reflection equation algebra, connected to a Hecke type R-matrix. Up to some reasonable additional conditions, the R-matrix is arbitrary (not necessary originating from quantum groups). We suggest a universal method for constructing finite dimensional irreducible representations in the framework of the Weyl approach well known in the representation theory of classical Lie groups and algebras. With this method a series of irreducible modules is constructed. The modules are parametrized by Young diagrams. The spectrum of central elements $s_k = \mathrm{Tr}_q L^k$ is calculated in the single-row and single-column representations. A rule for the decomposition of the tensor product of modules into a direct sum of irreducible components is also suggested.

PACS numbers: 02.10.Ud, 02.20.Uw

1. Reflection equation algebra

The reflection equation and the corresponding algebra, which will be called reflection equation algebra (REA for short), play a significant role in the theory of integrable systems and non-commutative geometry. In application to integrable systems the reflection equation with a spectral parameter is mainly used. It first appears in the work of Cherednik [1]. Usually it comprises the information about the behaviour of a system at a boundary—for example, it may describe the reflection of particles at a boundary of the configuration space.

The reflection equation without a spectral parameter is important for non-commutative geometry. One of the first applications that the corresponding REA found was in the theory of differential calculus on quantum groups (see, e.g., [2]). In such a differential calculus the REA with the Hecke type R-matrix is a non-commutative analogue of the algebra of vector fields on the groups $GL(N)$ or $SL(N)$. Besides this, the REA serves as a basis for a definition of quantum analogues of homogeneous spaces—orbits of the coadjoint representation of a Lie group, as well as quantum analogues of linear bundles over such orbits (see, e.g., [3, 4]).

In this paper we turn to problems of the representation theory of the REA without a spectral parameter. We are interested in the following main topics:

(i) a construction of finite dimensional irreducible representations and the calculation of the spectrum (characters) of central elements in these representations;
(ii) a rule for the decomposition of the tensor product of irreducible modules into irreducible components.

Before reviewing the known results, we introduce some necessary definitions and notation.

Consider an associative algebra $\mathcal{L}_q$ with the unit element $e_{\mathcal{L}}$ over the complex field $\mathbb{C}$ generated by n^2 elements $\hat{l}_i^j$, $1 \leqslant i, j \leqslant n$, n being a fixed positive integer. Let the generators satisfy the following quadratic commutation relations:

$$R_{12}\hat{L}_1R_{12}\hat{L}_1 - \hat{L}_1R_{12}\hat{L}_1R_{12} = 0 \qquad \hat{L}_1 \equiv \hat{L} \otimes I \tag{1.1}$$

where the matrix $\hat{L} \in \mathrm{Mat}_n(\mathcal{L}_q)$ is composed of $\hat{l}_i^j$: $\hat{L} = \|\hat{l}_i^j\|$. Here the lower index enumerates the rows while the upper one enumerates the columns. In (1.1) and everywhere below, use is made of compact matrix notation [5] when the index of an object indicates the vector space to which the object belongs (or in which this object acts). The symbol I stands for the identity matrix whose dimension is always clear from the context of the formulae. A numerical $n^2 \times n^2$ matrix R is a solution of *the Yang–Baxter equation*

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{1.2}$$

The algebra $\mathcal{L}_q$ described above will be called *the reflection equation algebra* (REA).

Impose now several additional conditions on the matrix R. The first of these is *the Hecke condition*

$$(R - qI)(R + q^{-1}I) = 0. \tag{1.3}$$

The parameter q is a fixed non-zero complex number with the only constraint being

$$q^k \neq 1 \qquad \forall\, k \in \mathbb{N}. \tag{1.4}$$

Note that all our subsequent constructions possess a well-defined 'classical limit' $q \to 1$. This limit corresponds to the REA with an involutive R-matrix: $R^2 = I$. A consequence of (1.4) is that the q-analogues of all integers are non-zero:

$$k_q \equiv \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0 \qquad \forall\, k \in \mathbb{N}. \tag{1.5}$$

As the second condition on the R-matrix, we shall suppose it to be *skew-invertible*; that is there exists an $n^2 \times n^2$ matrix Ψ such that

$$\sum_{a,b} R_{ia}^{jb}\Psi_{bk}^{as} = \delta_i^s\delta_k^j = \sum_{a,b} \Psi_{ia}^{jb}R_{bk}^{as}.$$

In compact notation, the above formula reads

$$\mathrm{Tr}_{(2)}R_{12}\Psi_{23} = P_{13} = \mathrm{Tr}_{(2)}\Psi_{12}R_{23} \tag{1.6}$$

where the symbol $\mathrm{Tr}_{(2)}$ means the calculation of the trace in the second space and P is the permutation matrix.

To formulate the last requirement on R one should consider the connection of the A_k series Hecke algebras with the group algebras of finite symmetric groups. One of the simplest definitions of the Hecke algebra reads as follows.

Fix a non-zero complex number q. The *Hecke algebra of the A_k series* ($k \geqslant 2$) is an associative algebra $H_k(q)$ over the complex field $\mathbb{C}$ generated by the unit element 1_H and $k-1$ generators σ_i subject to the following relations:

$$\left.\begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if} \quad |i-j| \geqslant 2 \\ (\sigma_i - q1_H)(\sigma_i + q^{-1}1_H) = 0 & \end{array}\right\} \qquad i = 1, 2, \ldots, k-1.$$

In some cases it proves to be convenient to consider q as a formal parameter and consider the Hecke algebra over the field of rational functions in the indeterminate q. We shall always bear in mind this extension when considering the classical limit $q \to 1$.

Let us treat R as the matrix of a linear operator (in a fixed basis) which acts in the tensor square $V^{\otimes 2}$ of a finite dimensional vector space V, with $\dim V = n$. Then an arbitrary Hecke R-matrix defines *the local representation* ρ_R of $H_k(q)$ in $V^{\otimes k}$:

$$\sigma_i \mapsto \rho_R(\sigma_i) = R_{i\,i+1} = I^{\otimes(i-1)} \otimes R \otimes I^{\otimes(k-i-1)} \in \mathrm{End}(V^{\otimes k}). \tag{1.7}$$

If the parameter q satisfies (1.4), then for any positive integer k the Hecke algebra $H_k(q)$ is known to be isomorphic to the group algebra $\mathbb{C}[\mathcal{S}_k]$ of the kth-order permutation group $\mathcal{S}_k$. As a consequence, there exist elements $\mathcal{Y}_{\nu(a)}(\sigma) \in H_k(q)$ which are the q-analogues of the Young idempotents (projectors) widely used in the theory of symmetric groups. These q-idempotents are parametrized by standard Young tableaux $\nu(a)$ corresponding to each diagram or equivalently to each partition $\nu \vdash k$. The number of all standard tableaux $\nu(a)$ which one can construct for a given ν will be denoted as $\dim[\nu]$:

$$\dim[\nu] = \#\{\nu(a)\}.$$

In the local representation (1.7) the elements $\mathcal{Y}_\nu(\sigma)$ are realized as some projector operators in $V^{\otimes k}$. With respect to the action of these projectors, the space $V^{\otimes k}$ is decomposed into a direct sum of subspaces V_ν as in the case of the symmetric group:

$$V^{\otimes k} = \bigoplus_{\nu\vdash k} \bigoplus_{a=1}^{\dim[\nu]} V_{\nu(a)} \qquad V_{\nu(a)} = Y_{\nu(a)}(R) \triangleright V^{\otimes k}. \tag{1.8}$$

The projector $Y_{\nu(a)}(R) = \rho_R(\mathcal{Y}_{\nu(a)})$ is given by some polynomial in matrices $R_{i\,i+1}$. The detailed treatment of these questions and explicit formulae for q-projectors can be found in the review [6]. There one can also find an extensive list of original papers; some of them are given in [7] for the reader's convenience.

So, we shall assume that there exists an integer $p > 0$ such that the image of the q-antisymmetrizer $\mathcal{A}^{(p+1)}(\sigma) \in H_k(q)$ ($\forall k > p$) in the local R-matrix representation ρ_R is identically zero, while the image of the q-antisymmetrizer $\mathcal{A}^{(p)}(\sigma) \in H_k(q)$ is a unit rank projector in the space $V^{\otimes k}$:

$$\exists p \in \mathbb{N}: \quad \begin{cases} \mathcal{A}^{(p+1)}(\sigma) \mapsto A^{(p+1)}(R) \equiv 0 & \\ \mathcal{A}^{(p)}(\sigma) \mapsto A^{(p)}(R) & \quad \mathrm{rank}\, A^{(p)}(R) = 1. \end{cases} \tag{1.9}$$

Such a number p will be called *the symmetry rank* of the matrix R. For example, the symmetry rank of the R-matrix connected to the quantum universal enveloping algebra $U_q(sl_n)$ is equal to n. Examples of $n^2 \times n^2$ R-matrices with $p < n$ (for $n \geqslant 3$) can be found in [8].

Introduce now two $n \times n$ matrices B and C:

$$B_1 = \mathrm{Tr}_{(2)}\Psi_{21} \qquad C_1 = \mathrm{Tr}_{(2)}\Psi_{12} \tag{1.10}$$

where Ψ is defined in (1.6). If the R-matrix has the symmetry rank p, these matrices are non-singular and their product is a multiple of the identity matrix [8]:

$$B \cdot C = \frac{1}{q^{2p}} I. \tag{1.11}$$

Besides this, B and C have the following traces:

$$\mathrm{Tr}\, B = \mathrm{Tr}\, C = \frac{p_q}{q^p}. \tag{1.12}$$

The matrices B and C play a central role in what follows.

The simplest example of the REA is obtained by choosing the $U_q(sl_2)$ R-matrix ($n = 2$):

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad \lambda \equiv q - q^{-1} \qquad \hat{L} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}.$$

In this case equation (1.1) leads to six permutation relations for the generators of the REA:

$$\begin{array}{ll} q^2\hat{a}\hat{b} = \hat{b}\hat{a} & q(\hat{b}\hat{c} - \hat{c}\hat{b}) = \lambda\hat{a}(\hat{d} - \hat{a}) \\ q^2\hat{c}\hat{a} = \hat{a}\hat{c} & q(\hat{c}\hat{d} - \hat{d}\hat{c}) = \lambda\hat{c}\hat{a} \\ \hat{a}\hat{d} = \hat{d}\hat{a} & q(\hat{d}\hat{b} - \hat{b}\hat{d}) = \lambda\hat{a}\hat{b}. \end{array} \tag{1.13}$$

Consider a map $\mathrm{Tr}_q \colon \mathrm{Mat}_n(\mathcal{L}_q) \to \mathcal{L}_q$ which is called *the quantum trace* [5]:

$$\mathrm{Tr}_q(X) \stackrel{\mathrm{def}}{=} \mathrm{Tr}(C \cdot X) \qquad X \in \mathrm{Mat}_n(\mathcal{L}_q). \tag{1.14}$$

One can show that the quantities

$$s_0(\hat{L}) = e_{\mathcal{L}} \qquad s_m(\hat{L}) = \mathrm{Tr}_q(\hat{L}^m) \quad 1 \leqslant m \leqslant p - 1 \tag{1.15}$$

are *independent* central elements of the REA (see [5]). Presumably these elements generate the whole centre of the REA, but we do not have a proof of this hypothesis. Calculation of the spectrum of central elements s_m in irreducible representations of $\mathcal{L}_q$ is one of our aims.

There is rather a lot of work devoted to the representation theory of the REA. First of all, this algebra possesses a large number of one-dimensional representations. For example, it is evident that at any choice of the R-matrix relation (1.1) will be satisfied if one sets $\hat{l}_i^j = \alpha\delta_i^j$. A less trivial representation can be obtained for our simple example (1.13) by putting $\hat{a} = 0$. Then the remaining generators are represented by three arbitrary complex numbers. Such representations were considered in detail in [9] for the REA with R-matrices coming from $U_q(sl_n)$ and its supersymmetric generalizations. The classification of all one-dimensional representations of the REA with the $U_q(sl_n)R$-matrix is presented in [10].

But since the REA is a non-commutative algebra, the images of a part of its generators are inevitably zero in any one-dimensional representation. That is, the kernel of any one-dimensional representation of the REA must contain some of its generators. These representations are not encompassed by our approach and we shall not consider them.

The main object of interest to us will be finite dimensional modules over the REA in which the linear operators representing REA generators are non-trivial and not mutually commutative. An example of such a representation can be constructed in the following way. Let us use the fact that the REA (1.1) is an adjoint comodule over some Hopf algebra which is similar to the algebra of functions over the quantum group. Suppose the commutations among the generators t_i^j of the Hopf algebra to be given by the matrix relation [5]

$$R_{12}T_1T_2 = T_1T_2R_{12}. \tag{1.16}$$

The above multiplication is compatible with the comultiplication Δ:

$$t_i^j \xrightarrow{\Delta} \sum_k t_i^k \otimes t_k^j. \tag{1.17}$$

Then (1.1) is covariant with respect to the transformation [11]

$$\hat{l}_i^j \to t_i^k S\big(t_p^j\big) \otimes \hat{l}_k^p \tag{1.18}$$

where $S(\cdot)$ stands for the antipodal map. Note that it is skew-invertibility (1.6) which allows one to define the antipode in a (bi)algebra (1.16) (to be more precise, in some extension; see [5]).

So, if one knows the representations of a Hopf algebra (1.16) then, given a representation of the REA, one can construct another one on the basis of (1.18).

Moreover, in the case of the $U_q(sl_n)R$-matrix one has an additional possibility of constructing non-trivial multidimensional representations. The fact is that in such a case there exists an embedding of the REA into $U_q(gl_n)$. At the level of generators this embedding is described by the formula [5]

$$\hat{L} = S(L^-)L^+ \tag{1.19}$$

where $L^{\pm}$ are matrices composed of the $U_q(gl_n)$ generators. Therefore, starting from a representation of the quantum group one can find the corresponding representation of the REA by means of (1.19). In recent paper [12] this approach was extended to the case of an arbitrary quasitriangular Hopf algebra. The authors of the cited paper constructed a universal solution of (1.1) based on the universal R-matrix of a quasitriangular Hopf algebra $\mathcal{H}$. The generators $\hat{l}_i^j$ turn out to be elements of the tensor product of $\mathcal{H}$ and its 'twisted dual' algebra.

However, it is worth pointing out that all the methods mentioned above are essentially based on the representation theory of objects which are external to the REA, namely, on the theory of quasitriangular Hopf algebras, the quantum groups being a particular case of these. But as was shown in [13], the Yang–Baxter equation possesses a lot of solutions not connected to a quantum group (see also [8]). For such solutions we cannot use map (1.19) and cannot construct the REA representations on the basis of quantum group ones.

Moreover, having fixed the R-matrix in (1.1), one completely defines all properties of the REA and its representation theory as well. The situation is similar to the Lie algebra theory, where the set of structure constants defines all properties of the algebra. Therefore, it is quite natural to develop the representation theory of the REA using only the given R-matrix—that is, entirely in terms of the REA itself.

The most efficient method for solving this problem seems to consist in the direct analysis of the explicit commutation relations of the REA generators. For the $U_q(sl_n)R$-matrix (at small values of n) one can proceed in analogy with the representation theory of the universal enveloping algebra of a (simple) Lie algebra. In this way, Kulish [14] succeeded in finding all highest vector representations of the simplest REA (1.13). Besides two one-dimensional representations, this algebra has a series of finite dimensional irreducible representations and a series of infinite dimensional representations—analogues of the Verma modules of the universal enveloping algebra.

Unfortunately, this approach is not universal. It has an essential dependence on the particular choice of the R-matrix. The explicit components of matrix relations (1.1) may become completely different when we change the R-matrix. Therefore, one would have to repeat the analysis of commutation relations from the very beginning for each possible R-matrix. Another obstacle is more technical. The fact is that even in the case of the $U_q(sl_n)R$-matrix the complexity of the explicit form of (1.1) increases very quickly with growing n. This leads to additional difficulties as compared with the case of a universal enveloping algebra, when the commutation relations among generators can be written in a compact form for an arbitrary n.

In the present paper we suggest a universal method for constructing finite dimensional representations of the REA generated by (1.1). These representations are parametrized by Young diagrams and exist for *any* R-matrix satisfying the additional conditions (1.3), (1.6) and (1.9). For the representations corresponding to single-row and single-column diagrams we calculate the spectrum of central elements (1.15). In the particular case of the $U_q(sl_2)$ R-matrix our result reproduces the series of finite dimensional representations of the REA (1.13) obtained in [14].

The paper is organized as follows. In section 2 we construct an irreducible representation of the REA with an arbitrary Hecke R-matrix possessing a finite symmetry rank. This representation is called the fundamental one (of B type) since its tensor products are decomposed into irreducible components similarly to those of the fundamental vector representation of $U(gl_n)$.

In section 3 we study the kth tensor power of the fundamental module of B type and consider its decomposition into higher dimensional REA modules.

Section 4 is devoted to another fundamental module (of R type). The construction is based on the general theory of dual Hopf algebras and can be easily generalized to the case of an arbitrary Hecke R-matrix. The connection of B and R type fundamental modules is established. We also consider an example of a reducible non-decomposable module over the REA which is not equivalent to either a B or R type module. At the end of the section we give a short summary of the results obtained and mention some open questions relating to the suggested approach.

2. The fundamental module of *B* type

Consider the REA generated by relations (1.1) and carry out the linear shift of generators

$$l_i^j = \hat{l}_i^j + \frac{1}{\lambda}\delta_i^j e_{\mathcal{L}} \tag{2.1}$$

where $e_{\mathcal{L}}$ is the unit element of $\mathcal{L}_q$ and $\lambda = q - q^{-1}$. On taking into account the Hecke condition (1.3) one obtains the commutation relations for the new generators

$$R_{12}L_1R_{12}L_1 - L_1R_{12}L_1R_{12} = R_{12}L_1 - L_1R_{12}. \tag{2.2}$$

In what follows we shall call the algebra generated by (2.2) *the modified reflection equation algebra* (mREA) and retain the notation $\mathcal{L}_q$ for it. Note that unless $q = 1$, the mREA is isomorphic to the REA with relations (1.1). As a consequence, any representation of the mREA can be transformed into that of the REA and vice versa. Nevertheless, these algebras are different at the classical limit since isomorphism (2.1) is broken at $q \to 1$ by virtue of the singularity of λ^{-1}.

In the particular case of the $U_q(sl_n)R$-matrix the classical limit of (2.2) gives the commutation relations of the $U(gl_n)$ generators.

So, consider the mREA $\mathcal{L}_q$ generated by the unit element and n^2 generators l_i^j with commutation relations (2.2). Let us take an n-dimensional vector space V and fix an arbitrary basis of n vectors e_i, $1 \leqslant i \leqslant n$. Define a *linear* map $\pi : \mathcal{L}_q \to \mathrm{End}(V)$ in accordance with the rules

$$\begin{aligned} &\pi(e_{\mathcal{L}}) = \mathrm{id}_V \\ &\pi\left(l_i^j\right) \triangleright e_k = e_i B_k^j \\ &\pi\left(l_{i_1}^{j_1} \cdot l_{i_2}^{j_2} \cdot \dots \cdot l_{i_k}^{j_k}\right) = \pi\left(l_{i_1}^{j_1}\right) \cdot \pi\left(l_{i_2}^{j_2}\right) \cdot \dots \cdot \pi\left(l_{i_k}^{j_k}\right) \qquad \forall k \in \mathbb{N} \end{aligned} \tag{2.3}$$

where the matrix B is defined in (1.10) and id_V is the identity operator on V.

Proposition 1 [17]**.** *The linear map (2.3) defines an irreducible representation of $\mathcal{L}_q$ (2.2) in the space V. This will be called the fundamental module of B type.*

Proof. To prove that π realizes the representation of $\mathcal{L}_q$ one only needs to verify that operators (2.3) do satisfy (2.2). This can be easily done by a straightforward calculation. The only fact needed in this consists in the following simple consequence of (1.6) and (1.10):

$$\mathrm{Tr}_{(1)} B_1 R_{12} = I. \tag{2.4}$$

Irreducibility follows from the non-singularity of B (1.11). Using this fact one can show that the operators $\pi\big(l_i^j\big)$ span $\mathrm{End}(V)$ and, therefore, the space V does not contain proper invariant subspaces with respect to π. □

If we make the shift (2.1) in our example (1.13) we get the commutation relations of the corresponding mREA:

$$\begin{array}{ll} q^2ab - ba = qb & q(bc - cb) = (\lambda a - 1)(d - a) \\ q^2ca - ac = qc & q(cd - dc) = c(\lambda a - 1) \\ ad = da & q(db - bd) = (\lambda a - 1)b. \end{array} \tag{2.5}$$

Here in the right-hand column the number 1 stands for the unit element $e_{\mathcal{L}}$. The matrices B and C have the forms

$$B = \begin{pmatrix} q^{-1} & 0 \\ 0 & q^{-3} \end{pmatrix} \qquad C = \begin{pmatrix} q^{-3} & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

The fundamental representation (2.3) reads

$$\pi(a) = \begin{pmatrix} q^{-1} & 0 \\ 0 & 0 \end{pmatrix} \qquad \pi(b) = \begin{pmatrix} 0 & q^{-3} \\ 0 & 0 \end{pmatrix} \qquad \pi(c) = \begin{pmatrix} 0 & 0 \\ q^{-1} & 0 \end{pmatrix} \qquad \pi(d) = \begin{pmatrix} 0 & 0 \\ 0 & q^{-3} \end{pmatrix}. \tag{2.6}$$

Given a representation of $\mathcal{L}_q$ one can find the corresponding representation of the quotient algebra:

$$\mathcal{SL}_q = \mathcal{L}_q / \{\mathrm{Tr}_q L\} \tag{2.7}$$

where $\{X\}$ stands for the ideal generated by a given subset $X \subset \mathcal{L}_q$. The commutation relations among the generators f_i^j of $\mathcal{SL}_q$ have the same form (2.2) as those of $\mathcal{L}_q$ (with the substitution $L \to F$ where $F = \| f_i^j \|$), but now the generators are linearly dependent because $\mathrm{Tr}_q F = 0$.

At $q \to 1$ in the case of the $U_q(sl_n) R$-matrix the commutation relations of the $\mathcal{SL}_q$ generators transform into those of the $U(sl_n)$ generators. For this reason the passage from $\mathcal{L}_q$ to $\mathcal{SL}_q$ (or from the $\mathcal{L}_q$ representation to the corresponding $\mathcal{SL}_q$ one) will be loosely called the *sl*-reduction in what follows.

The transformation of an irreducible $\mathcal{L}_q$ representation ρ acting in a finite dimensional space V into the $\mathcal{SL}_q$ representation $\bar{\rho}$ is realized as follows. Since $\mathrm{Tr}_q L$ is a central element of $\mathcal{L}_q$ and ρ is an irreducible representation, one gets

$$\rho(\mathrm{Tr}_q L) = \chi(\mathrm{Tr}_q L)\, \mathrm{id}_V \equiv \chi_1\, \mathrm{id}_V$$

where $\chi : Z(\mathcal{L}_q) \to \mathbb{C}$ is a character of the centre $Z(\mathcal{L}_q)$. Then straightforward calculation shows that the $\mathcal{SL}_q$ generators f_i^j in the representation $\bar{\rho}$ are given by

$$\bar{\rho}\big(f_i^j\big) = \frac{1}{\omega}\left(\rho\big(l_i^j\big) - \delta_i^j \frac{\chi_1}{\mathrm{Tr}\, C}\, \mathrm{id}_V\right) \qquad \omega = 1 - \lambda \frac{\chi_1}{\mathrm{Tr}\, C}. \tag{2.8}$$

The tracelessness property $\bar{\rho}(\mathrm{Tr}_q F) = 0$ is evident and the factor ω^{-1} ensures the correct normalization of the right-hand side of (2.2).

Remark 1. As can be easily seen from definition (1.1), the REA admits the 'renormalization' automorphism $\hat{l}_i^j \to z\hat{l}_i^j$ with non-zero complex number z. The same is true for the REA representations as well. At the level of the mREA representations this automorphism reads

$$\rho\big(l_i^j\big) \to \rho_z\big(l_i^j\big) = z\rho\big(l_i^j\big) + \delta_i^j \frac{1-z}{\lambda}\,\mathrm{id}_V \tag{2.9}$$

where ρ is an arbitrary mREA representation in the space V. On the basis of (2.8) one can show that the corresponding $\mathcal{SL}_q$ representation $\bar{\rho}$ does not depend on z; that is, the whole class of mREA representations ρ_z connected by the renormalization automorphism (2.9) give the same $\mathcal{SL}_q$ representation $\bar{\rho}$.

Let us now obtain the *sl*-reduction of the B type representation π defined by (2.3). By virtue of (1.11) one finds

$$\chi_1 = \chi(\mathrm{Tr}_q\, L) = q^{-2p}.$$

Then, taking into account (1.12) and (2.8), we find the B type representation $\bar{\pi}$ of algebra (2.7):

$$\bar{\pi}\big(f_i^j\big) = \frac{1}{\omega}\left(\pi\big(l_i^j\big) - \frac{\delta_i^j}{q^p p_q}\,\mathrm{id}_V\right) \qquad \omega = \frac{q^{1-p}}{p_q}(q^{p-2}(p+1)_q - 1). \tag{2.10}$$

Consider again our example (2.5) of the mREA $\mathcal{L}_q$. To get the corresponding algebra $\mathcal{SL}_q$ it is necessary to take the quotient of $\mathcal{L}_q$ over the ideal generated by $\mathrm{Tr}_q\, L$. Using the explicit form of C one has

$$\mathrm{Tr}_q\, L = \frac{1}{q^3}a + \frac{1}{q}d.$$

With a new generator $h = a - d$ one can rewrite the commutation relations for $\mathcal{SL}_q$ in terms of three independent quantities b, c and h:

$$\begin{aligned} q^2hb - bh &= 2_q b \\ hc - q^2ch &= -2_q c \\ q(bc - cb) &= h\left(1 - \frac{q\lambda}{2_q}h\right). \end{aligned} \tag{2.11}$$

Note that at $q \to 1$, relations (2.11) transform into the well known commutation relations for the generators of the universal enveloping algebra $U(sl_2)$.

Starting from (2.6) we have, due to (2.10),

$$\bar{\pi}(h) = \xi\begin{pmatrix} q & 0 \\ 0 & -q^{-1}\end{pmatrix} \qquad \bar{\pi}(b) = \xi\begin{pmatrix} 0 & q^{-1} \\ 0 & 0\end{pmatrix} \qquad \bar{\pi}(c) = \xi\begin{pmatrix} 0 & 0 \\ q & 0\end{pmatrix} \qquad \xi = \frac{q^2+1}{q^4+1}.$$

At the classical limit this representation turns into the fundamental vector representation of $U(sl_2)$.

Let us point out that at $q \neq \pm 1$ the usual trace of $\bar{\pi}(h)$ is not equal to zero (this property is restored at the classical limit only). However, the quantum trace of the matrix $\bar{\pi}(h)$ is equal to zero:

$$\mathrm{Tr}(C \cdot \bar{\pi}(h)) = 0.$$

It should be emphasized that in the above relation the matrix C is used to deform the trace of *operators* of the $\mathcal{SL}_q$ representation, while in (1.14) the quantum trace is taken in $\mathrm{Mat}_n(\mathcal{SL}_q)$ (or in $\mathrm{Mat}_n(\mathcal{L}_q)$).

So, in order to retain the tracelessness property of the $\mathcal{SL}_q$ representation in the space V one has to modify the definition of the operator trace in $\mathrm{End}(V) : \mathrm{Tr} \to \mathrm{Tr}_q$. This is no mere chance. The usual tensor category, like that of modules over $U(sl_n)$, is not suitable for use as the representation category for the algebra $\mathcal{L}_q$ (or $\mathcal{SL}_q$). The natural representation category for the above-mentioned algebras is some *quasitensor Schur–Weyl category* (the notion of the quasitensor category was introduced in [15]). It is the quantum trace (not the usual one) which turns out to be a natural morphism closely connected to the structure of the Schur–Weyl category. The detailed description of the category, the role of the quantum trace in it and the connection with the REA are considered in [16] and [17].

To complete the section, we clarify the connection for the B type representation (2.6) of the mREA (2.5) with the result of [14]. Applying the inverse linear shift of generators (see (2.1)) to representation (2.6) one gets the representation of the REA with quadratic relations (1.13). The matrices thus obtained are equivalent (connected by a similarity transform) to the matrices of the two-dimensional representation derived in [14]. This representation is the lowest one in a series of finite dimensional irreducible representations of (1.13). The higher dimensional modules of this series are equivalent to the q-symmetrical tensor powers of the B type modules. Their construction is considered in the next section. So, in the particular case of the $U_q(sl_2)\,R$-matrix our approach gives the known result of [14].

3. Higher dimensional modules of B type

Let us turn to the problem of the tensor product of the mREA modules. We are mainly interested in the following questions. Firstly, given the fundamental module V of B type, how do we define an mREA module structure in the tensor power $V^{\otimes k}$? Secondly, into which irreducible higher dimensional modules can one decompose the module $V^{\otimes k}$? Lastly, how can one decompose the tensor product of arbitrary higher dimensional modules (not only fundamental ones) into a sum of irreducible components? Here we propose answers to these questions.

3.1. The tensor product of fundamental modules

When trying to define the mREA module structure on the tensor product of fundamental modules one finds a serious difficulty. The problem is that it is not known whether the mREA (2.2) (as well as the REA (1.1)) possesses a *bialgebra* structure. As a consequence, in algebra (2.2) one cannot define the coproduct operation.

To clarify the importance of this operation, consider the case of the universal enveloping algebra $\mathcal{U} = U(sl_n)$ and consider particularly the definition of the $\mathcal{U}$ module structure on the tensor product of fundamental modules in the framework of the Weyl approach [18].

The algebra $\mathcal{U}$ is a bialgebra with the cocommutative coproduct

$$\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}.$$

As is known, a Lie algebra can always be embedded into its universal enveloping algebra. Let x be the image of a Lie algebra generator under such an embedding. The action of Δ on a (Lie) generator x of $\mathcal{U}$ is given by a simple formula:

$$\Delta(x) = x \otimes e_{\mathcal{U}} + e_{\mathcal{U}} \otimes x \stackrel{\text{def}}{=} x_{(1)} + x_{(2)}$$

where $e_{\mathcal{U}}$ is the unit element of $\mathcal{U}$.

Take now some irreducible representation $\rho : \mathcal{U} \to \mathrm{End}(V)$ in a finite dimensional vector space V. To get a representation of $\mathcal{U}$ in $V^{\otimes k}$ one first constructs a homomorphism $\Delta^{(k-1)} : \mathcal{U} \to \mathcal{U}^{\otimes k}$ by multiple application of the coproduct Δ:

$$\Delta^{(1)} \equiv \Delta \qquad \Delta^{(m)} = (\Delta \otimes \mathrm{id}_{(m-1)}) \circ \Delta^{(m-1)} \qquad \forall\, m \geqslant 2.$$

Then the image of $\mathcal{U}$ under such a homomorphism is represented in $V^{\otimes k}$ by the map $\rho^{\otimes k}$. For a given generator x of $\mathcal{U}$ these two steps can be written in the explicit form

$$\begin{aligned} &\text{(i)} \quad \Delta^{(k-1)} : x \mapsto \mathbf{x} = x_{(1)} + x_{(2)} + \cdots + x_{(k)} \in \mathcal{U}^{\otimes k} \\ &\text{(ii)} \quad \rho^{\otimes k} : \mathbf{x} \mapsto \rho^{\otimes k}(\mathbf{x}) \in \mathrm{End}(V^{\otimes k}). \end{aligned} \tag{3.1}$$

This representation is reducible. To extract the irreducible components one uses the fact that in $V^{\otimes k}$ it is possible to define a natural representation of the group algebra $\mathbb{C}[\mathcal{S}_k]$ of the kth-order permutation group $\mathcal{S}_k$. With respect to this representation the space $V^{\otimes k}$ is decomposed into a direct sum of irreducible $\mathbb{C}[\mathcal{S}_k]$ modules $V_{\nu(a)}$. The modules are parametrized by the standard Young tableaux $\nu(a)$, corresponding to all possible partitions $\nu \vdash k$. With respect to the representation of $\mathcal{U}$ the subspaces $V_{\nu(a)}$ are also irreducible. A generator x of $\mathcal{U}$ is represented in $V_{\nu(a)}$ by the following linear operator:

$$\rho_{\nu(a)}(x) = P_{\nu(a)}\rho^{\otimes k}(\mathbf{x})P_{\nu(a)} \tag{3.2}$$

where $P_{\nu(a)}$ is the Young projector in $V^{\otimes k}$ corresponding to the tableau $\nu(a)$. The modules parametrized by different tableaux of the same partition ν are equivalent.

Return now to the case of the mREA. As was already mentioned, this algebra does not possess a coproduct and for constructing the tensor product of mREA modules we cannot use the above scheme as it stands. But it proves to be possible to generalize the final formulae (3.1) and (3.2) to the case of the mREA.

Introduce the useful notation for a chain of R-matrices

$$R_i \equiv R_{ii+1} \qquad R^{\pm 1}_{(i\to j)} \stackrel{\text{def}}{=} \begin{cases} R_i^{\pm 1}R_{i+1}^{\pm 1}\cdots R_j^{\pm 1} & \text{if} \quad i < j \\ R_i^{\pm 1}R_{i-1}^{\pm 1}\cdots R_j^{\pm 1} & \text{if} \quad i > j \\ R_i^{\pm 1} & \text{if} \quad i = j. \end{cases} \tag{3.3}$$

The analogue of the reducible representation (3.1) is established in the following proposition.

Proposition 2. *Consider the fundamental $\mathcal{L}_q$ module V defined in proposition 1 and fix a basis e_i, $1 \leqslant i \leqslant n$, in V. The tensor product $V^{\otimes k}$ is also an $\mathcal{L}_q$ module. In the basis $e_{i_1} \otimes \cdots \otimes e_{i_k}$ of $V^{\otimes k}$ the matrices of operators representing the $\mathcal{L}_q$ generators are as follows:*

$$\rho_k^t\big(l_i^j\big) = \pi^t\big(l_i^j\big) \otimes I^{\otimes(k-1)} + \sum_{s=1}^{k-1} R^{-1}_{(s\to 1)}\big[\pi^t\big(l_i^j\big) \otimes I^{\otimes(k-1)}\big]R^{-1}_{(1\to s)}. \tag{3.4}$$

Here ρ_k^t and π^t stand for the transposed matrices.

Proof. The proposition is proved by direct calculations. Since the calculations are rather lengthy we shall not reproduce them in full detail, giving instead the list of important intermediate steps with the corresponding results.

We prove the proposition by induction on k. For $k = 1$ our assertion reduces to proposition 1 and hence is true. Suppose it to be true up to some $k-1$ and prove that then it is true for k.

One should verify that operators (3.4) do satisfy the commutation relations (2.2). Let us assign the number $k+1$ to the auxiliary space of indices of the $\mathcal{L}_q$ generators. One has to show that

$$R_{k+1}\rho_k(L_{k+1})R_{k+1}\rho_k(L_{k+1}) - \rho_k(L_{k+1})R_{k+1}\rho_k(L_{k+1})R_{k+1} = R_{k+1}\rho_k(L_{k+1}) - \rho_k(L_{k+1})R_{k+1}. \tag{3.5}$$

It is convenient to carry out the transposition of the matrices ρ in the above relation and to take into account that in accordance with (2.3) the matrix $\pi\big(l_i^j\big)$ reads

$$\pi^t(L_{k+1})_1 = P_{1k+1}B_{k+1}$$

where the identity enumerates the matrix indices of the representation space. Then, on substituting (3.4) into (3.5) we find that the first summand in the left-hand side decomposes into a sum of k^2 terms, a typical one being as follows:

$$R_{k+1}R^{-1}_{(n\to 1)}P_{1k+1}B_{k+1}\big[\mathrm{Tr}_{(1)}R^{-1}_{(1\to n)}R^{-1}_{(m\to 1)}B_1R_{1k+2}\big]R^{-1}_{(1\to m)} \equiv R_{k+1}Q(n,m)$$

where the last equality is the definition of $Q(n,m)$, $0 \leqslant n, m \leqslant k-1$. Here in $Q(0,m)$ the identity matrices are substituted for the chains $R^{-1}_{(1\to n)}$ and $R^{-1}_{(n\to 1)}$. The second summand in the left-hand side of (3.5) expands in a similar way but R_{k+1} stands on the right of Q.

By virtue of the supposition of the induction we conclude that it is sufficient to consider just $2k-1$ terms in each summand containing $Q(k-1,n)$ and $Q(n,k-1)$. So, one needs to examine the following expression in the left-hand side of (3.5):

$$\sum_{n=0}^{k-2}(R_{k+1}(Q(n,k-1)+Q(k-1,n)) - (Q(n,k-1)+Q(k-1,n))R_{k+1}) \\ + R_{k+1}Q(k-1,k-1) - Q(k-1,k-1)R_{k+1}. \tag{3.6}$$

The proposition will be proved if one can show that this is equal to

$$R_{k+1}R^{-1}_{(k-1\to 1)}P_{1k+1}B_{k+1}R^{-1}_{(1\to k-1)} - R^{-1}_{(k-1\to 1)}P_{1k+1}B_{k+1}R^{-1}_{(1\to k-1)}R_{k+1}. \tag{3.7}$$

Consider first the difference, containing $Q(n,k-1)$, $0 \leqslant n \leqslant k-2$. Since (see the appendix)

$$\mathrm{Tr}_{(1)}\, R^{-1}_{(1\to n)}R^{-1}_{(k-1\to 1)}B_1R_{1k+2} = R^{-1}_{(k-1\to 2)}P_{2k+2}B_{k+2}R^{-1}_{(2\to n+1)} \qquad \forall n \leqslant k-2$$

and (see, e.g., [19])

$$R_{12}B_1B_2 = B_1B_2R_{12} \tag{3.8}$$

we find

$$R_{k+1}(Q(n,k-1)+Q(k-1,n)) - (Q(n,k-1)+Q(k-1,n))R_{k+1} \\ = \lambda R^{-1}_{(n\to 1)}R^{-1}_{(k-1\to 1)}P_{1k+1}B_{k+1}P_{2k+2}B_{k+2}R^{-1}_{(2\to k-1)}R^{-1}_{(1\to n)} \\ - \lambda R^{-1}_{(n\to 1)}R^{-1}_{(k-1\to 2)}P_{1k+1}B_{k+1}P_{2k+2}B_{k+2}R^{-1}_{(1\to k-1)}R^{-1}_{(1\to n)}. \tag{3.9}$$

The calculation of the difference from $Q(k-1,k-1)$ in (3.6) is more involved. The trace contained in $Q(k-1,k-1)$ is as follows (see the appendix):

$$\mathrm{Tr}_{(1)}\, R^{-1}_{(1\to k-1)}R^{-1}_{(k-1\to 1)}B_1R_{1k+2} = I - \lambda P_{2k+2}B_{k+2} - \lambda\sum_{n=2}^{k-1}R^{-1}_{(n\to 2)}P_{2k+2}B_{k+2}R^{-1}_{(2\to n)}.$$

Finally, it is a matter of straightforward calculation to show that the unit matrix I in the above expression leads to the necessary contribution (3.7) while the other terms exactly cancel the unwanted summands of type (3.9). □

3.2. Decomposition of $V^{\otimes k}$ into mREA submodules

Our next goal is to find the decomposition of the $\mathcal{L}_q$ module $V^{\otimes k}$ into irreducible submodules, that is, to find an analogue of the classical formula (3.2). This can be done on the basis of the isomorphism $H_k(q) \cong \mathbb{C}[\mathcal{S}_k]$ which was discussed in the first section. For our construction the most important consequence of such an isomorphism is the existence of q-analogues of primitive Young idempotents $\mathcal{Y}_\nu(\sigma) \in H_k(q)$ and the decomposition (1.8) of $V^{\otimes k}$ into a direct sum of $H_k(q)$ modules.

The main result of this section is formulated in the following proposition.

Proposition 3. *Consider the mREA $\mathcal{L}_q$ generated by (2.2) with the Hecke R-matrix possessing the symmetry rank p (see (1.3), (1.6) and (1.9)). Let V be the fundamental $\mathcal{L}_q$ module of B type with a fixed basis e_i, $1 \leqslant i \leqslant n$. According to proposition 2 the space $V^{\otimes k}$ is also an $\mathcal{L}_q$ module for any positive integer k. Decompose the tensor product $V^{\otimes k}$ into the direct sum (1.8).*

Then each component $V_{\nu(a)}$ of the direct sum is an $\mathcal{L}_q$ module and the generators l_i^j are represented by linear operators $\hat{\pi}_{\nu(a)}\big(l_i^j\big) \in \mathrm{End}(V_{\nu(a)}) \hookrightarrow \mathrm{End}(V^{\otimes k})$. The matrices of these operators in the basis $e_{i_1} \otimes \cdots \otimes e_{i_k}$ of $V^{\otimes k}$ are of the form

$$\pi^t_{\nu(a)}\big(l_i^j\big) = Y_{\nu(a)}(R)\rho_k^t\big(l_i^j\big)Y_{\nu(a)}(R) \tag{3.10}$$

where ρ_k is defined in (3.4) of proposition 2 and the symbol t means matrix transposition.

The modules parametrized by different tableaux of the same partition $\nu \vdash k$ are equivalent.

Proof. Consider matrices (3.4) of the $\mathcal{L}_q$ representation in $V^{\otimes k}$. Let us assign the number $k+1$ to the auxiliary space of indices of $\mathcal{L}_q$ generators and rewrite the commutation relations in the form (3.5). The projectors $Y_{\nu(a)}(R)$ in (3.10) are some polynomials in matrices R_i, $1 \leqslant i \leqslant k-1$. The proof of the proposition is based on the fact that the matrix $\rho_k^t(L_{k+1})$ commutes with R_i for $1 \leqslant i \leqslant k-1$. Indeed, taking this for granted for the moment, we conclude that $\rho_k^t(L_{k+1})$ commutes with all projectors $Y_{\nu(a)}(R)$. Then, due to the orthogonality and the completeness of the set of projectors

$$\begin{aligned} &Y_{\nu(a)}(R)Y_{\mu(b)}(R) = \delta_{\nu\mu}\delta_{ab}Y_{\mu(b)}(R) \\ &\sum_{\nu\vdash k}\sum_{a=1}^{\dim[\nu]} Y_{\nu(a)}(R) = \mathrm{id}_{V^{\otimes k}} \end{aligned} \tag{3.11}$$

it is easy to see that relation (3.5) allows the projection onto each component $V_{\nu(a)}$ of the direct sum (1.8) and the matrices of the corresponding representation are given by (3.10).

To prove the commutativity supposed above one needs no particular property of $\pi\big(l_i^j\big)$ (an arbitrary $n\times n$ matrix X can be substituted for $\pi(l_i^j)$). The proof consists in a simple calculation and is completely based on the Yang–Baxter equation (1.2) and the Hecke condition (1.3).

The equivalence of modules parametrized by different Young tableaux of the same partition ν stems from the connection of the corresponding Young idempotents. It can be shown [6] that any two idempotents $\mathcal{Y}_{\nu(a)}(\sigma)$ and $\mathcal{Y}_{\nu(b)}(\sigma)$ are connected by a similarity transform. That is, there exists an *invertible* element $\mathcal{F}_{ab}(\nu|\sigma) \in H_k(q)$ which is a polynomial in σ_i such that

$$\mathcal{F}_{ab}(\nu|\sigma)\mathcal{Y}_{\nu(a)}(\sigma)\mathcal{F}^{-1}_{ab}(\nu|\sigma) = \mathcal{Y}_{\nu(b)}(\sigma) \qquad 1 \leqslant a, b \leqslant \dim[\nu].$$

In the local representation (1.7) of $H_k(q)$ in $V^{\otimes k}$ the element $\mathcal{F}_{ab}(\nu|\sigma)$ turns into the intertwining operator $F_{ab}(R)$ which ensures the equivalence of $\pi_{\nu(a)}$ and $\pi_{\nu(b)}$:

$$F_{ab}\pi^t_{\nu(a)}F^{-1}_{ab} = F_{ab}Y_{\nu(a)}\rho_k^t Y_{\nu(a)}F^{-1}_{ab} = Y_{\nu(b)}\rho_k^t Y_{\nu(b)} = \pi^t_{\nu(b)}.$$

The second equality in the above chain of transformations is valid due to the commutativity of ρ_k^t and $F_{ab}(R)$ since the latter is a polynomial in R_i, $1 \leqslant i \leqslant k-1$. □

A natural question arises about the irreducibility of the modules $V_{\nu(a)}$. For reasons discussed at the end of the paper, we suppose the modules to be irreducible, but we still have no general proof of this fact. We state it as a quite plausible hypothesis.

For the representations parametrized by single-row and single-column diagrams the formulae become much simpler and in this case one can explicitly calculate the spectrum of central elements (1.15).

Corollary 3.1. *Consider the $\mathcal{L}_q$ modules $V_{\nu(a)}$, defined in proposition 3.*

(i) For partitions $\nu = (k)$ and $\nu = (1^k)$ (for $k \leqslant p$) the matrices of operators representing the $\mathcal{L}_q$ generators are given by

$$\pi^t_{(k)}\big(l_i^j\big) = q^{1-k} k_q S^{(k)}(R)\big[\pi^t\big(l_i^j\big) \otimes I^{\otimes(k-1)}\big] S^{(k)}(R) \tag{3.12}$$

$$\pi^t_{[k]}\big(l_i^j\big) = q^{k-1} k_q A^{(k)}(R)\big[\pi^t\big(l_i^j\big) \otimes I^{\otimes(k-1)}\big] A^{(k)}(R) \qquad k \leqslant p \tag{3.13}$$

where $S^{(k)}$ and $A^{(k)}$ are the q-symmetrizer and the q-antisymmetrizer respectively.

(ii) In the representations $\pi_{(k)}$ and $\pi_{[k]}$ the spectrum χ of central elements $s_m = \mathrm{Tr}_q L^m$ takes the following values:

$$\chi_{(k)}(s_m) = q^{-m(p+k-1)-p} k_q (p+k-1)_q^{m-1} \tag{3.14}$$

$$\chi_{[k]}(s_m) = q^{-m(p-k+1)-p} k_q (p-k+1)_q^{m-1}. \tag{3.15}$$

Proof. The assertion (i) is the direct consequence of (3.10). Indeed, taking into account the relations

$$\begin{array}{ll} R_i^{\pm 1} S^{(k)}_{12\ldots k} = q^{\pm 1} S^{(k)}_{12\ldots k} = S^{(k)}_{12\ldots k} R_i^{\pm 1} & \\ & 1 \leqslant i \leqslant k-1 \\ R_i^{\pm 1} A^{(k)}_{12\ldots k} = -q^{\mp 1} A^{(k)}_{12\ldots k} = A^{(k)}_{12\ldots k} R_i^{\pm 1} & \end{array}$$

and the definition (1.5) of a q-number, one immediately gets (3.12) and (3.13) from (3.10) where the arbitrary projector $Y_{\nu(a)}$ should be replaced by $S^{(k)}(R)$ or $A^{(k)}(R)$ respectively.

In order to find the values of the characters (3.14) and (3.15) we consider $\pi^t_{(k)}(\mathrm{Tr}_q L^m)$ (take the q-symmetrical case for definiteness) and show that this is a multiple of the q-symmetrizer (the identity operator on the subspace $V_{(k)}$). The factor is equal to the character $\chi_{(k)}(s_m)$.

Taking into account (1.11) one rewrites the matrix involved in the form

$$\pi^t_{(k)}(\mathrm{Tr}_q L^m) = q^{m(1-k)-2p} k_q^m S^{(k)}_{12\ldots k}\big[\mathrm{Tr}_{(1)} B_1 S^{(k)}_{12\ldots k}\big]^{m-1} S^{(k)}_{12\ldots k}.$$

To calculate the trace $\mathrm{Tr}_{(1)} B_1 S^{(k)}_{12\ldots k}$ we use the recurrence relations for the q-(anti)symmetrizers (see, e.g., [8]):

$$S^{(1)}(R) = I \qquad S^{(k)}_{12\ldots k}(R) = \frac{1}{k_q} S^{(k-1)}_{2\ldots k}(R)(q^{1-k} I + (k-1)_q R_{12}) S^{(k-1)}_{2\ldots k}(R) \tag{3.16}$$

$$A^{(1)}(R) = I \qquad A^{(k)}_{12\ldots k}(R) = \frac{1}{k_q} A^{(k-1)}_{2\ldots k}(R)(q^{k-1} I - (k-1)_q R_{12}) A^{(k-1)}_{2\ldots k}(R) \tag{3.17}$$

where $S^{(k-1)}_{2\ldots k}$ is the q-symmetrizer of the $(k-1)$th order acting on the components of $V^{\otimes k}$ with indices 2 to k. Then using (2.4) and (1.12) one finds

$$\mathrm{Tr}_{(1)} B_1 S^{(k)}_{12\ldots k} = q^{-p} \frac{(p+k-1)_q}{k_q} S^{(k-1)}_{2\ldots k}.$$

Having found the above trace, one gets

$$\pi^t_{(k)}(\mathrm{Tr}_q L^m) = q^{-m(p+k-1)-p} k_q (p+k-1)_q^{m-1} S^{(k)}(R) = \chi_{(k)}(s_m)\, \mathrm{id}_{V_{(k)}} \tag{3.18}$$

which proves (3.14). □

It is worth pointing out that in the above formulae the central role belongs to the symmetry rank p of the R-matrix and not to the parameter n defining the size of the R-matrix and the number of generators in the corresponding REA $\mathcal{L}_q$. This feature is characteristic of the Schur–Weyl category connected to the series of representations considered, π_ν [16].

3.3. The sl-reduction

Our next aim is to explore a problem of the tensor product of fundamental $\mathcal{SL}_q$ representations (2.10). Actually, the solution for the problem follows from proposition 3 and (2.8). The only thing we need is the spectrum of the element $\mathrm{Tr}_q\, L$ in representations (3.10).

Lemma 1. *Let the partition $\nu \vdash k$ be of height s, that is,*

$$\nu = (\nu_1, \nu_2, \ldots, \nu_s) \qquad \sum_{r=1}^{s} \nu_i = k \qquad \nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_s > 0.$$

Then the spectrum of the central element $s_1 = \mathrm{Tr}_q\, L$ in the representation $\pi_{\nu(a)}$ $1 \leqslant a \leqslant \dim[\nu]$ is as follows:

$$\chi_\nu(s_1) = q^{-2p} \sum_{r=1}^{s} q^{2r-1-\nu_r} (\nu_r)_q \tag{3.19}$$

where p is the symmetry rank of the R-matrix and $(\nu_r)_q$ is the q-analogue of the integer ν_r (see definition (1.5)).

Proof. Let us find the matrix $\pi_{\nu(a)}(\mathrm{Tr}_q\, L)$ for an arbitrary $\nu(a)$. Using (1.14) and (1.11) one immediately gets from (3.10)

$$\pi^t_{\nu(a)}(\mathrm{Tr}_q\, L) = q^{-2p} Y_{\nu(a)}(R)\big(J_1^{-1} + J_2^{-1} + \cdots + J_k^{-1}\big) Y_{\nu(a)}(R).$$

Here the matrices J_k read

$$J_1 = I \qquad J_i = R_{(i-1\to 1)} R_{(1\to i-1)} \qquad i \geqslant 2. \tag{3.20}$$

They are the images of the *Jucys–Murphy* elements $\mathcal{J}_i(\sigma)$ of the Hecke algebra $H_k(q)$ under the local representation (1.7). The elements $\mathcal{J}_i(\sigma)$ generate the maximal commutative subalgebra in $H_k(q)$ and have the following important property (see, e.g., [6]):

$$\mathcal{J}_i(\sigma)\mathcal{Y}_{\nu(a)}(\sigma) = \mathcal{Y}_{\nu(a)}(\sigma)\mathcal{J}_i(\sigma) = q^{2(c_i - r_i)} \mathcal{Y}_{\nu(a)}(\sigma)$$

where c_i and r_i are respectively the coordinates of the column and the row to which the box with the index i belongs. The quantity $q^{2(c-r)}$ is called the *content* of the (c, r)th box of a given Young diagram. Here is an example of the diagram $\nu = (4, 3, 1^2)$ with the corresponding contents:

1	q^2	q^4	q^6
q^{-2}	1	q^2	
q^{-4}			
q^{-6}			

Therefore we come to the result

$$\pi^t_{\nu(a)}(\mathrm{Tr}_q\, L) = q^{-2p} \left(\sum_{i=1}^{k} q^{-2(c_i - r_i)} \right) Y_{\nu(a)} = \chi_\nu(\mathrm{Tr}_q\, L) Y_{\nu(a)}. \tag{3.21}$$

It is obvious that the sum of all contents (or their inverses) of a given tableau $\nu(a)$ is completely defined by the partition (diagram) ν and has the same value for all tableaux $\nu(a)$ corresponding to a given ν. As a consequence, the character $\chi_\nu(\mathrm{Tr}_q\, L)$ does not depend on a. Of course, this is in agreement with the equivalence of representations $\pi_{\nu(a)}$ parametrized by different tableaux $\nu(a)$ of the same partition ν.

It is a matter of a simple calculation to show that $\chi_\nu(\mathrm{Tr}_q\, L)$ represented as the sum of all inverse contents of the tableau $\nu_{(a)}$ can be written in the form (3.19). $\square$

It is interesting to note one peculiarity of the quantum case, namely, the spectrum of central elements distinguishes the representations more effectively than in the classical case. To show this, take q as a parameter and extend the complex field $\mathbb{C}$ to the field of rational functions in q. The character $\chi_\nu(s_1)$ can be identically transformed to the following expression:

$$q^{2p}\lambda\chi_\nu(s_1) = q^s s_q - \sum_{r=1}^{s} q^{2(r-\nu_r)-1} \equiv q^s s_q - \Omega(\nu_1, \ldots, \nu_s).$$

Given $\Omega(\nu_1, \ldots, \nu_s)$ as a function in q, one unambiguously restores the values of all ν_r, since the numbers $2(r - \nu_r)$ form a *strictly* increasing sequence. Therefore if the height s of a partition ν is fixed and known, then the spectrum of the first central element $s_1 = \mathrm{Tr}_q L$ is sufficient for distinguishing the representations. This is not so, of course, for the classical case $q \to 1$.

Now we are in a position to formulate the results for the higher dimensional modules of the algebra $\mathcal{SL}_q$.

Proposition 4. *Consider the algebra $\mathcal{SL}_q$ (2.7) with a Hecke R-matrix possessing the symmetry rank p. Let V be the fundamental $\mathcal{L}_q$ module of B type with a fixed basis e_i, $1 \leqslant i \leqslant n$. According to proposition 2 the tensor product $V^{\otimes k}$ is also an $\mathcal{L}_q$ module for any $k \in \mathbb{N}$. The following assertions are true.*

(i) The space $V^{\otimes k}$ is a reducible $\mathcal{SL}_q$ module. In the basis $e_{i_1} \otimes \cdots \otimes e_{i_k}$ the matrices of operators representing the $\mathcal{SL}_q$ generators f_i^j have the following form:

$$\bar{\rho}_k^t\big(f_i^j\big) = \frac{1}{\omega}\left(\rho_k^t\big(l_i^j\big) - \frac{\delta_i^j}{p_q q^p}\mathcal{Z}_k\right) \qquad \omega = \frac{q^{1-p}}{p_q}(q^{p-2}(p+1)_q - 1) \tag{3.22}$$

where the symbol t means matrix transposition, $\rho_k\big(l_i^j\big)$ is defined in (3.4) and $\mathcal{Z}_k$ is given by

$$\mathcal{Z}_k = I + \sum_{n=1}^{k-1} R^{-1}_{(n\to 1)} R^{-1}_{(1\to n)}.$$

(ii) Decompose the tensor product $V^{\otimes k}$ into the direct sum (1.8). Each component $V_{\nu(a)}$ of the direct sum is an $\mathcal{SL}_q$ submodule in $V^{\otimes k}$. The generators f_i^j are represented by linear operators with the following matrices:

$$\bar{\pi}^t_{\nu(a)}\big(f_i^j\big) = \frac{1}{\omega_\nu} Y_{\nu(a)}\left[\rho_k^t\big(l_i^j\big) - \delta_i^j \frac{q^p}{p_q}\chi_\nu(s_1) I^{\otimes k}\right] Y_{\nu(a)} \qquad \omega_\nu = 1 - \lambda\frac{q^p}{p_q}\chi_\nu(s_1) \tag{3.23}$$

the character $\chi_\nu(s_1)$ being defined in (3.19). The modules parametrized by different tableaux $\nu(a)$ of the same partition $\nu \vdash k$ are equivalent.

(iii) The spectrum $\bar{\chi}$ of the $\mathcal{SL}_q$ central elements $\bar{s}_m = \mathrm{Tr}_q\, F^m$ in representations $\bar{\pi}_{(k)}$ and $\bar{\pi}_{[k]}$ corresponding to $\nu = (k)$ and $\nu = (1^k)$ takes the following values:

$$\bar{\chi}_{(k)}(\bar{s}_m) = q^{-p-m}\frac{k_q(p-1)_q(p+k)_q}{(p+k-1)_q}\,\frac{\left[(p-1)_q^{m-1}(p+k)_q^{m-1} + (-1)^m k_q^{m-1}\right]}{(q^{p-2}(p+k)_q - k_q)^m} \tag{3.24}$$

$$\bar{\chi}_{[k]}(\bar{s}_m) = q^{-p+m}\frac{k_q(p+1)_q(p-k)_q}{(p-k+1)_q}\,\frac{\left[(p+1)_q^{m-1}(p-k)_q^{m-1} + (-1)^m k_q^{m-1}\right]}{(q^{p+2}(p-k)_q + k_q)^m}. \tag{3.25}$$

Proof. This proposition is a direct consequence of propositions 2 and 3, corollary 3.1 and rule (2.8). Indeed, the operators $\bar{\rho}_k(\mathrm{Tr}_q\, F)$ and $\bar{\pi}_{\nu(a)}(\mathrm{Tr}_q\, F)$ are obviously equal to zero. On the basis of propositions 2 and 3 one can verify that (3.22) and (3.23) satisfy (2.2) and that the factor ω_ν^{-1} ensures proper normalization of the right-hand side of (2.2).

As for the values (3.24) and (3.25), they can be found by straightforward but rather lengthy calculations on the basis of (3.23), (3.14) and (3.15). □

In the case of the $U_q(sl_n)R$-matrix we have $p = n$ and at the classical limit $q \to 1$ the spectrum (3.24), (3.25) of the $\mathcal{SL}_q$ central elements tends to the spectrum of the $U(sl_n)$ Casimir elements in the corresponding representations (see, e.g., [20]).

Finally, consider the tensor product of two (irreducible) modules V_μ and V_ν over $\mathcal{L}_q$ (or $\mathcal{SL}_q$) in which the representation operators π_μ and π_ν are given by (3.10) (or by (3.23)). Using the isomorphism $H_k \cong \mathbb{C}[\mathcal{S}_k]$ one can show (see, e.g., [16]) that the tensor product $V_\mu \otimes V_\nu$ is isomorphic to the following direct sum of $\mathcal{L}_q$ (or $\mathcal{SL}_q$) modules V_σ:

$$V_\mu \otimes V_\nu \cong c_{\mu\nu}^{\sigma} V_\sigma. \tag{3.26}$$

Here $c_{\mu\nu}^{\sigma}$ are the Littlewood–Richardson coefficients defining a ring structure in the set of Schur symmetric functions.

4. The fundamental module of *R* type

Besides the fundamental REA module of *B* type considered in the previous sections, one can construct another module which will be called *the fundamental module of R type*. In the case of the *R*-matrix connected to $U_q(sl_n)$ the corresponding representation originates from the general theory of dual Hopf algebras and we generalize it to the case of an arbitrary *R*-matrix.

4.1. The definition and tensor product decomposition rule

So, suppose at first that the *R*-matrix defining the structure of the REA is the Drinfel'd–Jimbo *R*-matrix connected to the quantum universal enveloping algebra $U_q(sl_n)$. In this case the Hopf algebra (1.16) is an algebra $\mathrm{Fun}_q(GL(n))$ of functions on the quantum group [5]. Besides this, there exists an embedding (1.19) of the corresponding REA $\mathcal{L}_q$ into $U_q(gl_n)$—the Hopf algebra dual to $\mathrm{Fun}_q(GL(n))$. As a consequence, it is possible to define a pairing among the generators $\hat{l}_i^{\,j}$ and $t_i^{\,j}$.

Using the explicit formulae for the pairing of $U_q(gl_n)$ generators $L^\pm$ and $\mathrm{Fun}_q(GL(n))$ generators T (see [5]), we get the following result:

$$\langle T_1 T_2 \cdots T_k, \hat{L}_{k+1}\rangle = R_{(k\to 1)} R_{(1\to k)} \equiv J_{k+1}. \tag{4.1}$$

Here we have used the compact notation (3.3) and (3.20) for the chains of *R*-matrices.

As is known from the Hopf algebra theory, any module over a Hopf algebra $\mathcal{H}$ can be transformed into a comodule over its dual Hopf algebra $\mathcal{H}^*$ and vice versa. Let us use this fact in order to define a representation of the REA $\mathcal{L}_q$ in a finite dimensional vector space V, $\dim V = n$.

On fixing a basis e_i, $1 \leqslant i \leqslant n$, one can convert the space V into a *left* comodule over the Hopf algebra $\mathcal{H} = \mathrm{Fun}_q(GL(n))$ by means of the corepresentation $\delta : V \to \mathcal{H} \otimes V$:

$$\delta(e_i) = t_i^{\,j} \otimes e_j \qquad \text{or} \qquad \delta(e_1) = T_1 \otimes e_1,$$

where the summation over the repeated indices is understood. Then in V we get a *right* $\mathcal{L}_q$ action by the well known rule

$$e_1 \triangleleft \hat{L}_2 = \langle T_1, \hat{L}_2\rangle e_1 = R_{12}^2 e_1.$$

This action can be easily extended to the tensor product $V^{\otimes k}$ for any $k \in \mathbb{N}$. Indeed the comodule structure $\delta_k : V^{\otimes k} \to \mathcal{H} \otimes V^{\otimes k}$ of $V^{\otimes k}$ is obvious:

$$\delta_k : e_1 \otimes \cdots \otimes e_k \mapsto T_1 \cdots T_k \otimes (e_1 \otimes \cdots \otimes e_k).$$

Hence, taking into account (4.1),

$$e_1 \otimes \cdots \otimes e_k \triangleleft \hat{L}_{k+1} = J_{k+1} e_1 \otimes \cdots \otimes e_k. \tag{4.2}$$

In the above relations we use the compact matrix notation where the index of a basis vector denotes the index of the factor in $V^{\otimes k}$ to which this vector belongs.

For the $U_q(sl_2) R$-matrix the representation (4.2) was obtained in [21] on the basis of (1.19), considering the connection of extended reflection algebras and the braid group on a handle-body.

It turns out that we can directly generalize (4.2) to the case of an *arbitrary* R-matrix. The following proposition is easy to verify.

Proposition 5. *Consider the REA $\mathcal{L}_q$ generated by relations (1.1) with an arbitrary R-matrix. The matrix will be treated as that of a linear operator acting in the tensor square of a finite dimensional vector space V, $\dim V = n$. Define a linear map $\theta_k : \mathcal{L}_q \to \mathrm{End}(V^{\otimes k})$ by the following rule:*

$$\begin{cases} \theta_k(e_{\mathcal{L}}) = \mathrm{id}_{V^{\otimes k}} \\ \theta_k(\hat{L}_{k+1}) = \alpha J_{k+1} \\ \theta_k\big(\hat{l}_{i_1}^{j_1} \cdot \hat{l}_{i_2}^{j_2} \cdot \cdots \cdot \hat{l}_{i_m}^{j_m}\big) = \theta_k\big(\hat{l}_{i_1}^{j_1}\big) \cdot \theta_k\big(\hat{l}_{i_2}^{j_2}\big) \cdot \cdots \cdot \theta_k\big(\hat{l}_{i_m}^{j_m}\big) \end{cases} \tag{4.3}$$

where $\alpha \neq 0$ is an arbitrary complex number. Then θ_k realizes a representation of $\mathcal{L}_q$ in the space $V^{\otimes k}$.

Proof. It is sufficient to substitute in the matrices $\theta_k(\hat{L}_{k+1}) = \alpha J_{k+1}$ in (1.1) rewritten in the form

$$R_{k+1} \hat{L}_{k+1} R_{k+1} \hat{L}_{k+1} - \hat{L}_{k+1} R_{k+1} \hat{L}_{k+1} R_{k+1} = 0$$

and make use of the following consequence of the Yang–Baxter equation (1.2):

$$\begin{aligned} (R_1 \cdots R_k) R_i &= R_{i+1} (R_1 \cdots R_k) \qquad 1 \leqslant i \leqslant k-1 \\ (R_k \cdots R_1) R_i &= R_{i-1} (R_k \cdots R_1) \qquad 2 \leqslant i \leqslant k. \end{aligned} \tag{4.4}$$

As for the numeric factor $\alpha \neq 0$, it can be arbitrary due to the renormalization automorphism $\hat{L} \to \alpha \hat{L}$ (see remark 1). □

Note that in proving proposition 5 we use nothing but the Yang–Baxter equation for the R-matrix. Therefore, representation (4.3) is valid not only for the quantum group R-matrix but also for an arbitrary solution of the Yang–Baxter equation (even one of non-Hecke type).

At $k > 1$ the representation θ_k is reducible. The Hecke condition (1.3) is needed for extracting the irreducible components of θ_k.

Proposition 6. *Let the REA $\mathcal{L}_q$ be generated by (1.1) with a Hecke R-matrix. Consider the representation θ_k (4.3) in the space $V^{\otimes k}$. Decompose $V^{\otimes k}$ into a direct sum of subspaces $V_{\nu(a)}$ in accordance with (1.8).*

Then each $V_{\nu(a)}$ is an $\mathcal{L}_q$ submodule and the matrices of the linear operators representing the generators $\hat{l}_i^j$ are given by

$$\theta_{\nu(a)}(\hat{L}_{k+1}) = Y_{\nu(a)}(R) \theta_k(\hat{L}_{k+1}) Y_{\nu(a)}(R). \tag{4.5}$$

The modules parametrized by different tableaux of the same partition $\nu \vdash k$ are equivalent.

Proof. The proof is based on the fact that the q-projectors $Y_\nu(R)$ are actually polynomials in $J_1, \dots, J_k$ for $\nu \vdash k$ (see [6]) and therefore all $Y_\nu(R)$ commute with R_{k+1}. Besides this, being the images of Jucys–Murphy elements, the operators J_i commute with J_{k+1}. Another way to verify this is by straightforward calculation on the basis of (4.4). As a result, the projectors $Y_\nu(R)$ commute with $\theta_k(\hat{L}_{k+1})$. Moreover, taking into account the orthonormal condition (3.11), we get

$$Y_{\nu(a)}\theta_k(\hat{L}_{k+1})Y_{\nu(a)} = Y_{\nu(a)}\theta_k(\hat{L}_{k+1}) = \theta_k(\hat{L}_{k+1})Y_{\nu(a)}.$$

The above relations lead to the following equality:

$$R_{k+1}Y_{\nu(a)}\theta_k Y_{\nu(a)}R_{k+1}Y_{\nu(a)}\theta_k Y_{\nu(a)} = Y_{\nu(a)}R_{k+1}\theta_k R_{k+1}\theta_k Y_{\nu(a)}.$$

As a consequence, the relation

$$R_{k+1}\theta_k(\hat{L}_{k+1})R_{k+1}\theta_k(\hat{L}_{k+1}) = \theta_k(\hat{L}_{k+1})R_{k+1}\theta_k(\hat{L}_{k+1})R_{k+1}$$

which applies in $\mathrm{End}(V^{\otimes k})$ admits projection into subspaces $\mathrm{End}(V_{\nu(a)})$:

$$R_{k+1}\theta_{\nu(a)}(\hat{L}_{k+1})R_{k+1}\theta_{\nu(a)}(\hat{L}_{k+1}) = \theta_{\nu(a)}(\hat{L}_{k+1})R_{k+1}\theta_{\nu(a)}(\hat{L}_{k+1})R_{k+1}.$$

The equivalence of $V_{\nu(a)}$ and $V_{\nu(b)}$ corresponding to different tableaux of the same partition ν is proved in the same way as in proposition 3. □

It is worth mentioning that for constructing representations θ_ν one does not need the symmetry rank of R to be finite.

As in the case of a B type module, one can explicitly calculate the spectrum of central elements (1.15) in the representations parametrized by single-row and single-column diagrams.

Corollary 6.1. *In the representations parametrized by partitions $\nu = (k)$ and $\nu = (1^k)$ the spectrum $\hat{\chi}$ of the central elements $s_m = \mathrm{Tr}_q\,\hat{L}^m$ takes the following values:*

$$\hat{\chi}_{(k)}(s_m) = q^{-p}\left(q^{-2m}p_q + \lambda\frac{(p+k)_q}{(k+1)_q}q^{m(k-1)}[m(k+1)]_q\right) \tag{4.6}$$

$$\hat{\chi}_{[k]}(s_m) = q^{-p}\left(q^{2m}p_q - \lambda\frac{(p-k)_q}{(k+1)_q}q^{-m(k-1)}[m(k+1)]_q\right) \qquad k \leqslant p. \tag{4.7}$$

Proof. We shall consider the case of a q-symmetric representation $\theta_{(k)}$ for definiteness. Let us first calculate $\theta_{(k)}(\mathrm{Tr}_q\,\hat{L})$. In accordance with (4.3) and (4.5) the matrices representing the REA generators read

$$\theta_{(k)}(\hat{L}_{k+1}) = S^{(k)}J_{k+1}S^{(k)}.$$

Taking into account (see [6]) that

$$S^{(k+1)} = S^{(k)}\frac{J_{k+1} - q^{-2}}{q^{2k} - q^{-2}} \tag{4.8}$$

we rewrite the matrix $\theta_{(k)}(\hat{L}_{k+1})$ in the equivalent form

$$\theta_{(k)}(\hat{L}_{k+1}) = \lambda q^{k-1}(k+1)_q S^{(k+1)} + q^{-2}S^{(k)}.$$

Next, by virtue of

$$\mathrm{Tr}_{q\,(k+1)}S^{(k+1)} = q^{-p}\frac{(p+k)_q}{(k+1)_q}S^{(k)}$$

we come to the final result

$$\theta_{(k)}(\mathrm{Tr}_q\,\hat{L}) = q^{-p}(q^{-2}p_q + \lambda q^{k-1}(p+k)_q)S^{(k)} \equiv \hat{\chi}_{(k)}(s_1)\,\mathrm{id}_{V_{(k)}}. \tag{4.9}$$

Then, on the basis of (4.8) and the relations

$$S^{(k+1)}S^{(k)} = S^{(k+1)} \qquad S^{(k+1)}J_{k+1} = q^{2k}S^{(k+1)}$$

one can prove (4.6) by induction on the power m of $\mathrm{Tr}_q\,\hat{L}^m$, where (4.9) serves as the first step. The final step of the induction gives

$$\hat{\chi}_{(k)}(s_m) = q^{-p}\left(q^{-2m}p_q + \lambda q^{m(k-1)}(p+k)_q\sum_{r=0}^{m-1}q^{(k+1)(2r+1-m)}\right).$$

With the substitution $t = q^{k+1}$ one can easily show that

$$\sum_{r=0}^{m-1}q^{(k+1)(2r+1-m)} = \frac{[m(k+1)]_q}{(k+1)_q}$$

coming thereby to the desired result (4.6). □

The representation θ_1 in the space V itself is irreducible and the matrices of operators representing $\mathcal{L}_q$ generators are as follows:

$$\theta_1(\hat{L}_2) = R_{12}^2 \tag{4.10}$$

where the indices of the first space are those of matrices from $\mathrm{End}(V)$ and the indices of the second space enumerate the generators of the algebra. The space V with the above $\mathcal{L}_q$ representation will be called *the fundamental module of R type*.

The representation of the mREA (2.2) obtained from (4.10) by the shift (2.1) reads

$$\theta_1(L_2) = -R_{12} \tag{4.11}$$

where we first perform a renormalization of (4.10) by the factor $\alpha = -\lambda^{-1}$.

4.2. The sl-reduction

In order to pass from the REA representation $\theta_{\nu(a)}$ (4.5) to the corresponding representation $\bar{\theta}_{\nu(a)}$ of the algebra $\mathcal{SL}_q$ we need to calculate the spectrum of $\theta_{\nu(a)}(\mathrm{Tr}_q\,\hat{L})$.

Lemma 2. *Let the partition $\nu \vdash k$ be of the height s, that is,*

$$\nu = (\nu_1, \nu_2, \ldots, \nu_s) \qquad \sum_{r=1}^{s}\nu_i = k \qquad \nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_s > 0.$$

Then the spectrum of the central element $s_1 = \mathrm{Tr}_q\,\hat{L}$ in the representation $\theta_{\nu(a)}$, $1 \leqslant a \leqslant \dim[\nu]$, is as follows:

$$\theta_{\nu(a)}(\mathrm{Tr}_q\,\hat{L}) = \zeta_\nu(s_1)Y_{\nu(a)} \qquad \zeta_\nu(s_1) = q^{-p}p_q + \lambda\sum_{r=1}^{s}q^{\nu_r+1-2r}(\nu_r)_q \tag{4.12}$$

where p is the symmetry rank of the R-matrix and $(\nu_r)_q$ is the q-analogue of the integer ν_r (see definition (1.5)).

Proof. The lemma is proved by straightforward calculation in analogy with the proof of lemma 1. □

Let us consider the representations $\theta_{(k)}$ and $\theta_{[k]}$ corresponding to single-row and single-column diagrams in more detail. In this case one can explicitly calculate the spectrum of the central elements similarly to the B type representation.

Proposition 7. *Let the Hecke type R-matrix have the symmetry rank p. Consider the REA representations of R type $\theta_{(k)}$ and $\theta_{[k]}$ parametrized by partitions $\nu = (k)$ and $\nu = (1^k)$ (4.5). Then the corresponding representations $\bar{\theta}$ of the $\mathcal{SL}_q$ generators $f_i^{\,j}$ are as follows:*

$$\bar{\theta}_{(k)}(F_{k+1}) = \frac{q^{1-p}(p+k)_q}{q^{2-p}(p+k)_q - k_q}\left(S^{(k)}I_{k+1} - \frac{p_q(k+1)_q}{(p+k)_q}S^{(k+1)}\right) \tag{4.13}$$

$$\bar{\theta}_{[k]}(F_{k+1}) = \frac{q^{-1-p}(p-k)_q}{q^{-2-p}(p-k)_q + k_q}\left(\frac{p_q(k+1)_q}{(p-k)_q}A^{(k+1)} - A^{(k)}I_{k+1}\right). \tag{4.14}$$

The spectrum $\bar{\zeta}$ of the $\mathcal{SL}_q$ central elements $\bar{s}_m = \mathrm{Tr}_q\, F^m$ in these representations takes the following values:

$$\bar{\zeta}_{(k)}(\bar{s}_m) = q^{-p-m(p-1)}\frac{k_q(p-1)_q(p+k)_q}{(k+1)_q}\frac{\left[(p+k)_q^{m-1} + (-1)^m k_q^{m-1}(p-1)_q^{m-1}\right]}{(q^{2-p}(p+k)_q - k_q)^m} \tag{4.15}$$

$$\bar{\zeta}_{[k]}(\bar{s}_m) = q^{-p-m(p+1)}\frac{k_q(p+1)_q(p-k)_q}{(k+1)_q}\frac{\left[(-1)^m(p-k)_q^{m-1} + k_q^{m-1}(p+1)_q^{m-1}\right]}{(q^{-2-p}(p-k)_q + k_q)^m}. \tag{4.16}$$

Proof. The proof consists in direct calculations on the basis of (2.8) and we shall not present it here. □

4.3. The interrelation between modules of B and R types

Let us now find a connection between the fundamental modules of B and R types. If the symmetry rank $p = 2$ (for example, when R stems from $U_q(sl_2)$), these modules are equivalent. To be more precise, the situation is as follows. By virtue of (1.9), the q-antisymmetrizer $A^{(2)}$ is a unit rank projector in $V^{\otimes 2}$ and its matrix can be written in the form

$$A_{i_1 i_2}^{j_1 j_2} = u_{i_1 i_2} v^{j_1 j_2}$$

the matrices $\|u_{ij}\|$ and $\|v^{ij}\|$ being non-singular. Then one can show that the representations π and θ_1 of the mREA (2.2) are connected by the relation

$$q^2 u_1 \cdot \pi(L_2) \cdot u_1^{-1} = qI_{12} + \theta_1(L_2).$$

After the *sl*-reduction we come to the representations $\bar{\pi}$ and $\bar{\theta}_1$ of the algebra $\mathcal{SL}_q$ (2.7) and simplify the above formula to the expression

$$u_1 \cdot \bar{\pi}(F_2) \cdot u_1^{-1} = \bar{\theta}_1(F_2),$$

$F = \|f_i^{\,j}\|$ being the matrix composed of the $\mathcal{SL}_q$ generators.

In the case $p > 2$, the fundamental modules of B and R types are not equivalent. Constraining ourselves to the case of $\mathcal{SL}_q$ algebra (2.7) we shall prove that the R type representation $\bar{\theta}(F)$ is equivalent to $\bar{\pi}_{[p-1]}(F)$ obtained from (3.13) by means of *sl*-reduction (3.23).

For this purpose, consider in more detail the structure of the subspace $V_{[p-1]} \subset V^{\otimes(p-1)}$. By definition (1.8), the subspace $V_{[p-1]}$ is the image of the q-antisymmetrizer $A^{(p-1)}$:

$$V_{[p-1]} = A^{(p-1)}(R) \triangleright V^{\otimes(p-1)}.$$

Since the symmetry rank of the R-matrix is equal to p, the q-antisymmetrizer $A^{(p)}$ is a unit rank projector and its matrix can be written in the form

$$A^{(p)}{}_{i_1\ldots i_p}^{j_1\ldots j_p} = u_{i_1\ldots i_p} v^{j_1\ldots j_p} \tag{4.17}$$

where as follows from (3.11) the tensors u and v are normalized by the condition

$$\sum_{\{i\}} u_{i_1\ldots i_p} v^{i_1\ldots i_p} = 1.$$

It is convenient to introduce the following linear combinations of the basis vectors of the space $V^{\otimes(p-1)}$:

$$\epsilon^i \stackrel{\text{def}}{=} \sum_{\{a\}} v^{ia_2\ldots a_p} e_{a_2} \otimes \cdots \otimes e_{a_p}. \tag{4.18}$$

The following lemma establishes an important property of the vectors ϵ^i.

Lemma 3. *Consider the set of n vectors $\epsilon^i \in V^{\otimes(p-1)}$ defined in (4.18). These are eigenvectors of the q-antisymmetrizer $A^{(p-1)}$ and they form a basis of the subspace $V_{[p-1]}$:*

$$A^{(p-1)}(R) \triangleright \epsilon^i = \epsilon^i \qquad \forall\, \mathbf{w} \in V_{[p-1]}: \quad \mathbf{w} = \sum_i w_i \epsilon^i. \tag{4.19}$$

Proof. Consider the recurrence relation (3.16) for the q-antisymmetrizer $A^{(p)}$ and calculate the trace in the first matrix space with the matrix B_1:

$$\mathrm{Tr}_{(1)}\, B_1 A^{(p)}_{12\ldots p} = \frac{1}{q^p p_q} A^{(p-1)}_{2\ldots p}.$$

By virtue of (4.17) we get the following expression for the matrix $\mathbb{A}$ of $A^{(p-1)}$:

$$\mathbb{A}_{a_2\ldots a_p}^{b_2\ldots b_p} = q^p p_q \sum_{m,n} B_m^n u_{na_2\ldots a_p} v^{mb_2\ldots b_p} \tag{4.20}$$

where for compactness we omit the superscript $(p-1)$ of the matrix $\mathbb{A}$. Also we need the formula connecting the matrix C (1.10) and the tensors u and v. It can be shown (see [8]) that

$$C_i^j = \frac{p_q}{q^p} \sum_{\{a\}} u_{ia_2\ldots a_p} v^{ja_2\ldots a_p} \equiv \frac{p_q}{q^p} \sum_{\{a\}} u_{i\{a\}} v^{j\{a\}} \tag{4.21}$$

where in the last equality we have introduced a convenient multi-index notation.

Finally, taking into account definition (4.18) we get the necessary result (the summation over repeated indices is understood)

$$\begin{aligned} A^{(p-1)} \triangleright \epsilon^i &= v^{ia_2\ldots a_p} \mathbb{A}_{a_2\ldots a_p}^{b_2\ldots b_p} e_{b_2} \otimes \cdots \otimes e_{b_p} \equiv v^{i\{a\}} \mathbb{A}_{\{a\}}^{\{b\}} \mathbf{e}_{\{b\}} \\ &= q^p p_q v^{i\{a\}} \mathbf{e}_{\{b\}} v^{m\{b\}} B_m^n u_{n\{a\}} = q^p p_q \epsilon^m B_m^n u_{n\{a\}} v^{i\{a\}} \\ &= q^{2p} \epsilon^m B_m^n C_n^i = \epsilon^i. \end{aligned}$$

Here, in the last step, we have used (1.11).

Therefore, under the action of the q-antisymmetrizer $A^{(p-1)}$ the space $W = \mathrm{Span}\{\epsilon^i\}$ is an invariant subspace in $V_{[p-1]}$ and hence $W = V_{[p-1]}$. But, as was proved in [8],

$$\dim V_{[p-1]} = \dim V = n.$$

Therefore the n vectors ϵ^i cannot be linearly dependent since if they are $\dim V_{[p-1]} = \dim W < n$. So, the set of eigenvectors ϵ^i of the q-antisymmetrizer $A^{(p-1)}$ can be taken as a basis of $V_{[p-1]}$. □

Now we are ready to establish the connection of B and R type fundamental modules in the case of an R-matrix with a finite symmetry rank.

Proposition 8. *Let the R-matrix have the symmetry rank p. Then the $\mathcal{SL}_q$ representation $\bar\theta_1$ obtained from (4.11) is equivalent to $\bar\pi_{[p-1]}$ obtained from (3.13) by sl-reduction (3.23).*

Proof. Let us first consider the $\mathcal{L}_q$ representation $\pi_{[p-1]}$ (3.13) which acts in the subspace $V_{[p-1]}$. By virtue of lemma 3 we shall find the matrices of operators $\pi_{[p-1]}\big(l_i^j\big)$ in the basis of vectors ϵ^k (4.18). Using (3.13) we obtain (in the same notation as in the proof of lemma 3)

$$\begin{aligned}\frac{q^{2-p}}{(p-1)_q}\pi_{[p-1]}\big(l_i^j\big)\triangleright\epsilon^k &= v^{k\{a\}}\mathbb{A}^{m\{c\}}_{\{a\}}B^j_m\mathbb{A}^{\{b\}}_{i\{c\}}\,\mathbf{e}_{\{b\}} = (\text{use }(4.20))\\ &= q^{2p}p_q^2\,\mathbf{e}_{\{b\}}v^{r\{b\}}B^l_s\big(v^{k\{a\}}u_{l\{a\}}\big)B^n_r\big(v^{sm\{c\}}u_{ni\{c\}}\big)B^j_m\\ &= q^{3p}p_q\epsilon^r\big(B^l_sC^k_l\big)B^n_r\big(v^{sm\{c\}}u_{ni\{c\}}\big)B^j_m = q^pp_q\epsilon^rB^n_r\big(u_{ni\{c\}}v^{km\{c\}}\big)B^j_m.\end{aligned}$$

Introduce an $n^2\times n^2$ matrix Ω with matrix elements

$$\Omega^{r_1r_2}_{s_1s_2} = p_q(p-1)_q\sum_{\{a\}}u_{s_1s_2\{a\}}v^{r_1r_2\{a\}}.$$

Then, the matrix of the operator $\pi_{[p-1]}\big(l_i^j\big)$ in the basis ϵ^k has the form (in compact notation)

$$(\pi_{[p-1]}(L_2))_1 = q^{2(p-1)}B_1\Omega_{12}B_2. \tag{4.22}$$

With the use of (3.16) for $A^{(p)}$ and (4.21) for C one can express the matrix Ω in a more explicit form. Omitting straightforward calculations we write down the final result:

$$\Omega_{12} = q^{2p-1}(C_1C_2 - qC_1\Psi_{21}C_1)$$

where Ψ is the skew-inverse to the R-matrix as defined in (1.6). Substituting this into (4.22) we find

$$(\pi_{[p-1]}(L_2))_1 = q^{-3}I_{12} - q^{2p-2}\Psi_{21}C_1B_2.$$

After *sl*-reduction (3.23) we get the representation of the $\mathcal{SL}_q$ algebra:

$$(\bar\pi_{[p-1]}(F_2))_1 = \frac{q^{1-p}}{(p-1)_q+q^{p+2}}\big(I_{12} - q^{3p}p_q\Psi_{21}C_1B_2\big).$$

The *sl*-reduction of the R type representation (4.11) leads in turn to the result

$$\bar\theta_1(F_2) = \frac{q^{p+1}}{(p-1)_q+q^{p+2}}\big(I_{12} - q^{-p}p_qR_{12}\big).$$

Next we take into account the connection of Ψ and R (see the appendix):

$$q^{2p}C_1\Psi_{21}B_2 = R^{-1}_{12}. \tag{4.23}$$

With this formula one immediately gets

$$C_1(\bar\pi_{[p-1]}(F_2))_1C_1^{-1} = \bar\theta_1(F_2),$$

which means that the corresponding modules are equivalent. □

4.4. Undecomposable modules: an example

It is natural to question the completeness of the set of representations thus obtained. In other words, is an arbitrary finite dimensional REA module equivalent to the direct sum of modules V_ν defined in proposition 3?

The answer to this question is negative. First of all, it is obviously not the case for the direct sums of specific one-dimensional modules described in [9, 10]. As we have already mentioned in section 1, these modules are beyond the scope of our approach. Secondly, the REA (1.1) possesses reducible finite dimensional but *undecomposable* modules. The corresponding mREA representations do not admit a finite classical limit $q \to 1$. Let us give a simple example of such a module for algebra $\mathcal{L}_q$ (1.13).

We start from the one-dimensional representation $\rho : \mathcal{L}_q \to \mathbb{C}$ (see [9]):

$$\rho(\hat{L}) = \begin{pmatrix} 0 & x \\ y & z \end{pmatrix} \qquad x, y, z \in \mathbb{C}. \tag{4.24}$$

Then we use the comodule property (1.18) in order to get the higher dimensional representation of $\mathcal{L}_q$. For this purpose take the known R-matrix representation γ of (1.16):

$$\gamma(T_1) = P_{12}R_{12} \qquad \gamma(S(T_1)) = R_{12}^{-1}P_{12}.$$

Here P is the transposition matrix, R is the $U_q(sl_2)$ R-matrix and the second matrix space stands for the representation space V, $\dim V = 2$. Then in accordance with (1.18) we construct a two-dimensional representation ρ_2 of $\mathcal{L}_q$ in the space $V \otimes \mathbb{C} \cong V$:

$$\rho_2(\hat{L}_1) = \gamma(T_1)\rho(\hat{L}_1)\gamma(S(T_1)) = R_{21}\rho(\hat{L}_2)R_{21}^{-1}.$$

For the $\mathcal{L}_q$ generators (1.13) the explicit form of the representation ρ_2 reads as follows:

$$\rho_2(\hat{a}) = \begin{pmatrix} 0 & -q\lambda x \\ 0 & 0 \end{pmatrix} \qquad \rho_2(\hat{b}) = \begin{pmatrix} qx & 0 \\ 0 & q^{-1}x \end{pmatrix}$$
$$\rho_2(\hat{c}) = \begin{pmatrix} q^{-1}y & -\lambda z \\ 0 & qy \end{pmatrix} \qquad \rho_2(\hat{d}) = \begin{pmatrix} z & q^{-1}\lambda x \\ 0 & z \end{pmatrix}.$$

The module V with the representation ρ_2 is reducible. The one-dimensional submodule is spanned by the basis vector e_1. The corresponding one-dimensional representation

$$\hat{a} \mapsto 0 \qquad \hat{b} \mapsto qx \qquad \hat{c} \mapsto q^{-1}y \qquad \hat{d} \mapsto z$$

is connected to the initial one (4.24) by an automorphism η of $\mathcal{L}_q$ [9]:

$$\eta \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} = \begin{pmatrix} \hat{a} & \omega\hat{b} \\ \omega^{-1}\hat{c} & \hat{d} \end{pmatrix} \qquad \forall \omega \in \mathbb{C}^{\times}.$$

Nevertheless, being reducible, the module V is obviously undecomposable, since matrices $\rho_2(\hat{a})$ and $\rho_2(\hat{d})$ cannot be transformed into diagonal form (unless $x = 0$). Therefore, this module cannot be presented as the direct sum of modules V_ν constructed in section 3.

So, examining the completeness of the set of V_ν, we have to reduce the class of admissible modules to completely reducible ones and reformulate the question in the following way: is any *completely reducible* finite dimensional module over the REA (1.1) isomorphic to a direct sum of modules V_ν?

For an arbitrary R-matrix with a finite symmetry rank we have no definite answer to this question. Given only the symmetry rank of R, one has too little information on the concrete structure of the corresponding REA. Perhaps an analysis of the explicit commutation relations is needed here. The question of irreducibility of the modules V_ν themselves is also open in this case.

As for the R-matrix originating from the quantum universal enveloping algebra $U_q(sl_n)$ $(p = n)$, it is highly plausible that the finite direct sums of the modules V_ν exhaust all finite dimensional completely reducible representations of the REA (of course, up to direct sums of one-dimensional modules from [9, 10]). This is based on the fact that the matrix elements of the corresponding representations are rational functions in q with a non-singular limit as $q \to 1$. At that limit the mREA $\mathcal{SL}_q$ (2.7) tends to the algebra $U(sl_n)$ and all the $\mathcal{SL}_q$ modules V_ν go to the corresponding modules over $U(sl_n)$. In particular, the modules V_ν described in proposition 3 must be irreducible.

To conclude, we briefly summarize the main results and discuss some open problems and perspectives.

For the reflection equation algebra we have constructed a series of finite dimensional representations which are parametrized by Young diagrams. The representations exist for any R-matrix satisfying the additional conditions (1.3), (1.6) and (1.9). The corresponding modules V_ν are simple objects of a quasitensor Schur–Weyl category described in detail in [16]. As follows from (3.26), the Grothendiek ring of the Schur–Weyl category for the Hecke R-matrix with the symmetry rank p is isomorphic to that of the category of finite dimensional modules over $U(sl_p)$. Nevertheless, the dimensions of the modules and the characters of the central elements could drastically differ from each other.

Also, it is worth mentioning some further problems in this approach. The first of these is the problem of constructing the representation theory for the REA connected to R-matrices of the Birman–Murakami–Wenzl type. Examples are given by R-matrices originating from the quantum groups of the B, C and D series. The key point here is developing an adequate technique for finding the q-analogues of the Young idempotents.

Another interesting problem is the representation theory for the REA with a spectral parameter. This could find wide application in the theory of integrable systems.

Acknowledgments

The author would like to thank D I Gurevich and P N Pyatov for valuable and helpful discussions on the problems considered in this paper. Also I express my gratitude to the staff and guests of the Max-Planck-Institut für Mathematik (Bonn) for kind hospitality, an encouraging and friendly atmosphere and perfect conditions for work.

Appendix

This is a technical section where some formulae auxiliary to the main text are proved. First, we prove the trace formulae which were used in proposition 2. The decisive role belongs to the following result.

Lemma 4. *Let R be a solution of the Yang–Baxter equation (1.2), satisfying the additional condition (1.6). Then*

$$\mathrm{Tr}_{(0)}\, B_0 R_{01} R_{02}^{-1} = P_{12} B_1 \tag{A.1}$$

where P is the transposition matrix and B is defined in (1.10).

Proof. Rewrite the Yang–Baxter equation (1.2) in the equivalent form

$$R_{12} R_{23} R_{12}^{-1} = R_{23}^{-1} R_{12} R_{23}.$$

Using this equation and definition (1.6) of the skew-inverse matrix Ψ we obtain the following relation:

$$\mathrm{Tr}_{(0)}\, \Psi_{10} R_{02} R_{03}^{-1} = P_{12}\, \mathrm{Tr}_{(0)}\, R_{10}^{-1} R_{20} \Psi_{03} P_{23}.$$

Now calculate the trace in the first space. Since $\mathrm{Tr}_{(1)}\, \Psi_{10} = B_0$, we get

$$\mathrm{Tr}_{(0)}\, B_0 R_{02} R_{03}^{-1} = \mathrm{Tr}_{(01)}\, R_{20}^{-1} P_{12} R_{20} \Psi_{03} P_{23} = \mathrm{Tr}_{(0)}\, \Psi_{03} P_{23} = B_3 P_{23} = P_{23} B_2.$$

This result differs from (A.1) only in the notation for the matrix spaces. □

So, we are ready to prove the trace formulae used in proposition 2.

(i) $\mathcal{T}(n, k-1) \equiv \mathrm{Tr}_{(1)}\, R^{-1}_{(1\to n)} R^{-1}_{(k-1\to 1)} B_1 R_{1k+2}$ at $n < k-1$. First of all, one should use (4.4) in order to draw the chain $R^{-1}_{(1\to n)}$ to the right of $R^{-1}_{(k-1\to 1)}$:

$$R^{-1}_{(1\to n)} R^{-1}_{(k-1\to 1)} = R^{-1}_{(k-1\to 1)} R^{-1}_{(2\to n+1)}.$$

The chain $R^{-1}_{(2\to n+1)}$ evidently commutes with $B_1 R_{1k+2}$; therefore,

$$\mathcal{T}(n, k-1) = R^{-1}_{(k-1\to 2)} \left[\mathrm{Tr}_{(1)}\, R^{-1}_{12} B_1 R_{1k+2}\right] R^{-1}_{(2\to n+1)}.$$

Using the cyclic property of the trace and then relation (A.1), we come to the desired result:

$$\mathcal{T}(n, k-1) = R^{-1}_{(k-1\to 2)} P_{2k+2} B_{k+2} R^{-1}_{(2\to n+1)}.$$

(ii) The calculation of $\mathcal{T}(k-1, k-1)$ is more cumbersome. The main difficulty is that in this case the chains of R-matrices cannot be drawn through each other. As a consequence, it is not so easy to decrease the number of R-matrices whose indices belong to the first space in order to apply (A.1). However, with the help of the Hecke condition and the Yang–Baxter equation the product of R-matrix chains contained in $\mathcal{T}(k-1, k-1)$ can be transformed as follows:

$$R^{-1}_{(1\to k-1)} R^{-1}_{(k-1\to 1)} = I - \lambda R^{-1}_1 - \lambda \sum_{n=2}^{k-1} R^{-1}_{(n\to 2)} R^{-1}_1 R^{-1}_{(2\to n)}.$$

The terms on the right-hand side contain at most one R-matrix with indices in the first space and hence, upon multiplying by $B_1 R_{1k+2}$, we can calculate $\mathrm{Tr}_{(1)}$ with the help of (A.1). As a result we get the formula which was used in the proof of proposition 2:

$$\mathcal{T}(k-1, k-1) = I - \lambda P_{2k+2} B_{k+2} - \lambda \sum_{n=2}^{k-1} R^{-1}_{(n\to 2)} P_{2k+2} B_{k+2} R^{-1}_{(2\to n)}.$$

(iii) Finally, relation (4.23) is a direct consequence of (A.1). Indeed, multiply (A.1) by Ψ_{13} from the right and take the trace in the first space. Due to the definition (1.6) of the matrix Ψ, we find

$$\mathrm{Tr}_{(0)}\, B_0 P_{03} R^{-1}_{02} = \mathrm{Tr}_{(1)}\, P_{12} B_1 \Psi_{13} = B_2 \Psi_{23}\, \mathrm{Tr}_{(1)}\, P_{12} = B_2 \Psi_{23}.$$

On the other hand, due to the cyclic property of the trace,

$$\mathrm{Tr}_{(0)}\, B_0 P_{03} R^{-1}_{02} = \mathrm{Tr}_{(0)}\, P_{03} R^{-1}_{02} B_0 = R^{-1}_{32} B_3\, \mathrm{Tr}_{(0)}\, P_{03} = R^{-1}_{32} B_3.$$

Therefore,

$$B_2 \Psi_{23} = R^{-1}_{32} B_3.$$

Multiplying this by C_3 from the right and using (1.11), we come to the relation

$$q^{2p} B_2 \Psi_{23} C_3 = R^{-1.}_{32}$$

Actually this is equivalent to (4.23), since on the basis of (3.8) one can easily show that

$$B_2 \Psi_{23} C_3 = C_3 \Psi_{23} B_2.$$

References

[1] Cherednik I V 1984 *Teor. Mat. Fiz.* **61** 35–44 (in Russian)
[2] Schupp P, Watts P and Zumino B 1992 *Lett. Mat. Phys.* **25** 139–47
Zumino B 1991 *Preprint* UCB-PTH-62/91
Isaev A P and Pyatov P 1993 *Phys. Lett.* A **179** 81–90
[3] Donin J and Mudrov A 2002 *Lett. Mat. Phys.* **62** 17–32
[4] Gurevich D and Saponov P 2002 *J. Phys. A: Math. Gen.* **35** 9629–43
Gurevich D and Saponov P 2001 *J. Phys. A: Math. Gen.* **34** 4553–69
[5] Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1989 *Alg. i Analiz* **1** 178–206 (in Russian)
Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1990 *Leningr. Math. J.* **1** 193–225 (Engl. Transl.)
[6] Ogievetsky O and Pyatov P 2000 *Proc. Int. Conf. on Symmetries and Integrable Systems (Dubna)* ed S Z Pakuliak (Dubna: JINR Publishing) pp 39–88
Ogievetsky O and Pyatov P 2000 *Preprint* CPT-2000/P.4076
[7] Gyoja A 1986 *Osaka J. Math.* **23** 841–52
Dipper R and James G 1987 *Proc. London Math. Soc.* **54** 57–82
Murphy G E 1992 *J. Algebra* **152** 492–513
[8] Gurevich D I 1990 *Alg. i Analiz* **2** 119 (in Russian)
Gurevich D I 1991 *Leningr. Math. J.* **2** 801–28 (Engl. Transl.)
[9] Kulish P P, Sasaki R and Schwiebert C 1993 *J. Math. Phys.* **34** 286–304
[10] Mudrov A 2002 *Lett. Math. Phys.* **60** 283-91
[11] Kulish P P and Sklyanin E K 1992 *J. Phys. A: Math. Gen.* **25** 5963-76
Kulish P P and Sasaki R 1993 *Progr. Theor. Phys.* **89** 741–61
[12] Donin J, Kulish P P and Mudrov A I 2003 *Lett. Math. Phys.* **62** 179–94
[13] Belavin A A and Drinfel'd V G 1982 *Func. Analiz i Pril.* **16** 1–29 (in Russian)
[14] Kulish P P 1994 *Alg. i Analiz* **6** 195–205 (in Russian)
Kulish P P 1995 *St. Petersburg Math. J.* **6** 365–74 (Engl. Transl.)
[15] Reshetikhin N Yu 1989 *Alg. i Analiz* **1** 169–88 (in Russian)
Reshetikhin N Yu 1990 *Leningr. Math. J.* **1** 491–513 (Engl. Transl.)
[16] Gurevich D, Leclercq R and Saponov P 2002 *J. Geom. Phys.* **44** 251–78
[17] Gurevich D, Leclercq R and Saponov P 2002 *Preprint* math.QA/0207268
[18] Weyl H 1973 *The Classical Groups, Their Invariants and Representations* (Princeton, NJ: Princeton University Press)
[19] Gurevich D, Pyatov P and Saponov P 1997 *Lett. Math. Phys.* **41** 255–64
[20] Barut A O and Rączka R 1986 *Theory of Group Representations and Applications* 2nd edn (Singapore: World Scientific)
[21] Schwiebert C 1994 *J. Math. Phys.* **35** 5288–305